Tutorial 2 - ALOHA

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Notice: If you find any mistakes, please open an issue at `https://github.com/robomarvin1501/notes_networking`

1 Introduction

2 Open-Systems Interconnect (OSI) model

The OSI model provides an abstract way of representing the functions needed in a communication system. It divides the networking tasks into separate layers. This model has 7, but sometimes there are 5, depending on how they are divided. Each layer uses the abstractions provided by the layer below it, and provides services to the one above it.

#	Layer Name	Data Unit	Function
1	Physical	Bit	Transmit and receive raw bits over some physical medium
2	Data link	Frame	Transmission of data frames between 2 nodes
3	Network	Packet	Structure and manage multi node network (addressing, routing, etc.)
4	Transport	Segment	Transmission (reliable or not) of segments between points in the network (ack'ing, etc.)
5	Session	Data	Manage a continuous communication session
6	Presentation	Data	Translation of data between a networking and an application (e.g. encryption / decryption)
7	application	Data	High level APIs between network applications (e.g. GET, POST)

Table 1: OSI 7 layer model

In this tutorial, and the next, we are going to focus on layer 2, but in the future, we will discuss all the layers (aside from 1, we can leave that for the physicists and the engineers).

Examples:

- Layer 7 (Application): DNS (turn website names to addresses), HTTP (send web pages)
- Layer 4 (Transport): TCP (reliably send segments), UDP (unreliably send segments)
- Layer 3 (Network): IP (sending and receiving of packets)
- Layer 2 (Data Link): MAC

3 Layer 2 - Data link

These operate over a share medium, where all nodes in the network broadcast and listen on the same channel. When a node broadcasts, it uses the full link bandwidth. If two nodes broadcast at the same time, there is a **collision**, and the data is lost. We ignore the noise on the channel.
Consider a protest. Only one person can talk at once, since when multiple people start talking, it all devolves into noise, and no information can be gathered (other than maybe lots of people have lots of things to say).

3.1 Definitions

Name	Explanation	Notation	Units
Bandwidth	The maximum rate of data transfer on the communication channel	B	$\frac{bit}{sec}$
Throughput	The fraction of the bandwidth used to send traffic	-	None
Goodput	The fraction of the bandwidth used to successfully send traffic	η	None

Table 2: Definitions table

Note that

$$\eta = \frac{\text{total average effective bandwidth usage}}{\text{total bandwidth}} = \frac{\text{total average effective time}}{\text{total time}}$$

So, assuming transmission uses the full bandwidth, $0 \leq \eta \leq 1$. An example of throughput vs goodput would be the background noise within a class. Throughput is the amount of noise made, but the goodput is how much information is communicated.

3.1.1 Example question

Let us assume that the link bandwidth is $B = 5$. Given a packet size of 30 bits, how long does it take to send a packet?

$$T = \frac{|\text{packet}|}{B} = \frac{30}{5} \cdot \frac{\text{bit}}{\frac{\text{bit}}{\text{second}}} = 6s$$

Assume that we need to transmit 3 packets, with a size of 30 bits, within 2 seconds. What is the required bandwidth?

$$B = \frac{|\text{packet}|}{B} = \frac{3 \cdot 30}{2} = 45 \frac{\text{bits}}{\text{sec}}$$

4 ALOHA

The ALOHA protocol is a protocol, originally developed in Hawaii for creating a local network that could span their islands. It is the first wireless network (undersea cabling was not good enough yet), and is the basis of the Ethernet, WiFi, and the 1G mobile network protocols. It makes the following assumptions:

- N nodes in the network. We usually analyse for $N \to \infty$
- B - link bandwidth
- L - number of bits in a frame
- Time is divided into slots of length $T = \frac{L}{B}$ T is the time required for a message to be transmitted. Successful transmissions use the entire bandwidth, and the entire time slow.
- A node can start transmitting *only* at the beginning of a slot. If a message arrives in the middle of a slot, then it will be sent at the beginning of the next slot
- **Collisions detection**: When a node *finishes* transmitting a message, it can detect collisions (if there are, it can then resend the message in the future)
- Propagation (physical delay) time is discarded

There are two main types of ALOHA that will be discussed, pure vs slotted:

- Pure ALOHA assumes that every node has its own clock (since synchronising clocks is an expensive task)
- Slotted ALOHA assumes that all nodes are *somehow* synchronised. We will not discuss how.

4.1 Binomial vs Poisson ALOHA

Let there be X_p, the number of transmission messages in a time segment T.

4.1.1 Binomial assumption

We assume that there are n end points, with probability p:

$$X_P \sim Bin\left(n, p\right)$$

4.1.2 Poisson assumption

There are ∞ end points, together with a rate of λ per T:

$$X_P \sim Poi\left(\lambda T\right)$$

Recall that if n is large, and p is small, then $Bin\left(n, p\right) \approx Poi\left(\lambda\right)$, where $np = \lambda$.

We're going to analyse 4 models, Binomial slotted, and pure ALOHA, and Poisson slotted, and pure ALOHA.

4.2 Binomial slotted

We assume that at any given time, all nodes have something to transmit, and each node tries to transmit in each slot with probability p_{send}. What is the goodput of this protocol? Note that each successful message effectively uses the entire bandwidth.

Since all the slots are the same, let us consider a single time slot. We need to find how much time (the expectation), was spent on *successful transmission* in this slot. If we define X_p as the number of transmissions in a **single** time slot, then we want to find

$$P_{suc} = \mathbb{P}\left[X_p = 1\right]$$

Where $X_p \sim Bin\left(n, p_{send}\right)$ The probability for a successful frame:

$$\begin{aligned} P_{bin} &= \binom{n}{x} p^x \left(1-p\right)^{n-x} \\ P_{suc} &= \mathbb{P}\left[X_p = 1\right] = np_{send}\left(1-p_{send}\right)^{n-1} \end{aligned}$$

Let us denote T_{suc} as the time spent on successful transmission in a single slot. What is $\mathbb{E}\left[T_{suc}\right]$?

$$\mathbb{E}\left[T_{suc}\right] = T \cdot P_{suc} + 0 \cdot \left(1 - P_{suc}\right) = T \cdot P_{suc}$$

and the goodput of the network will be

$$\eta = \frac{\mathbb{E}\left[T_{suc}\right]}{T} = P_{suc} = np_{send}\left(1-p_{send}\right)^{n-1}$$

To find the relation between successful, and unsuccessful slots. We can ask how many slots on average do we have before there is a successful slot? Let us define X_p as the number of tries. We will note that

$$X_p \sim Geo\left(p_{suc}\right) \wedge \mathbb{E}\left[X_p\right] = \frac{1}{p_{suc}}$$

So the goodput of the network is

$$\eta = \frac{T}{T \cdot \left(\frac{1}{P_{suc}}\right)} = P_{suc}$$

We want to maximise the goodput of η where:

$$\eta = \mathbb{P}\left[X_p\right] = 1 = np_{send}\left(1-p_{send}\right)^{n-1}$$

Let us denote $p = p_{send}$, and solve for p:

$$\begin{aligned} \frac{d}{dp} np\left(1-p\right)^{n-1} &= n \cdot \left(1-p\right)^{n-1} + np \cdot \left(n-1\right)\left(1-p\right)^{n-2} \cdot \left(-1\right) \\ &= n \cdot \left(1-p\right)^{n-2} \cdot \left[\left(1-p\right) - p \cdot \left(n-1\right)\right] \\ &\vdots \\ \implies \frac{d}{dp}\eta &= 0 \\ \implies p &= \frac{1}{n} \end{aligned}$$

We get that $p^* = \frac{1}{n}$ maximises η.

How good is p^*? Well, let us set p_{send} as above, and take $n \to \infty$. We want to maximise the goodput of η.

$$\begin{aligned} \lim_{n\to\infty} \eta\left(p^*\right) &= \lim_{n\to\infty} np^*\left(1-p^*\right)^{n-1} \\ &= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{n-1} \\ &= \frac{1}{e} \\ &\approx 0.37 \end{aligned}$$

Slotted ALOHA gives us freedom to fully distribute our system, but 0.37 is not very efficient.

4.3 Pure ALOHA

Here, each station transmits at the beginning of when *it* thinks the slot begins. Since the clocks are not synchronised, this can be at any time. Given a fixed packet length of T, then there is a vulnerable time period (when there can be a collision) of $2T$: For T before transmission begins, and T from when the packet begins. If there is a collision, then the transmitter waits a random amount of time before retransmitting.

What is the probability of a successful frame?

$$\begin{aligned} P_{suc} &= \mathbb{P}\left[\text{some node transmitted}\right] \cap \mathbb{P}\left[\text{no other node transmitted in 2 overlapping slots}\right] \\ &= \mathbb{P}\left[\text{some node transmitted}\right] \cdot \mathbb{P}\left[\text{no other node transmitted in 2 overlapping slots}\right] \\ &= np_{send}\left(\left(1-p_{send}\right)^2\right)^{n-1} \\ &= np_{send}\left(1-p_{send}\right)^{2(n-1)} \end{aligned}$$

Goodput: Let us define a random variable

$$T_{suc} = \begin{cases} T, & \text{successful slot} \\ 0, & \text{else} \end{cases}$$

Therefore $\mathbb{E}\left[T_{suc}\right] = T \cdot P_{suc}$, and therefore

$$\eta = \frac{\mathbb{E}\left[T_{suc}\right]}{T} = P_{suc} = np_{send}\left(1-p_{send}\right)^{2(n-1)}$$

The maximum lies in $p_{send} = \frac{1}{2n-1}$, and the goodput is $\frac{1}{2e}$, which is **half** the goodput of slotted ALOHA.

5 Questions

Exercise 1. *Host C and host D are connected in 2 different networks, the upper network runs Slotted ALOHA, and contains two more nodes, where the lower network runs pure ALOHA, and contains 3 more nodes. D never sends messages, and when C has a message to send to D, it chooses which network to use, and sends it with a probability of q. C chooses the upper network with a probability of p, and in every slot, each other node X sends with a probability of q.*

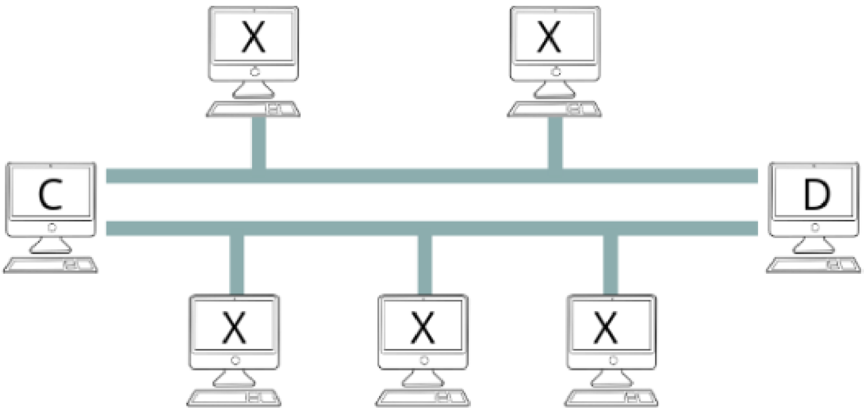

Figure 1: ALOHA network

C has one attempt to send the message to D, what is the probability that the message will arrive at D?

Solution. For the upper network, which is slotted,

$$\begin{aligned} P_{suc} &= \mathbb{P}\left[\text{C sends a message}\right] \cdot \mathbb{P}\left[\text{the others do not send}\right] \\ &= q \cdot (1-q)^2 \end{aligned}$$

For the lower network, which is pure

$$\begin{aligned} P_{suc} &= q \cdot \mathbb{P}\left[\text{No other transmission for two slots}\right] \\ &= q \cdot \mathbb{P}\left[\text{No other transmission for two slots}\right] \\ &= q \cdot \left((1-q)^2\right)^3 \\ &= q \cdot (1-q)^6 \end{aligned}$$

So

$$\begin{aligned} p_{suc}^{total} &= p \cdot p_{suc}^{upper} + (1-p)\, p_{suc}^{lower} \\ &= q\left(p\,(1-q)^2 + (1-p)\,(1-q)^6\right) \end{aligned}$$

□

Now C tries to send at the beginning of each slot. What is the goodput of the network?

Solution.

$$\begin{aligned} P_{suc}^{lower} &= \binom{3}{1} q\,(1-q)^{2+2} \\ &= 3q\,(10q)^4 \\ P_{suc}^{upper} &= \binom{2}{1} q\,(1-q) \\ \eta &= (1-p)\, P_{suc}^{lower} + p \cdot P_{suc}^{upper} \end{aligned}$$

□

Exercise 2. *There are $2n$ nodes in the network. The first n run pure ALOHA, with slots of length $2T$. The last n run slotted ALOHA with slots of length T. Every node sends a message at the beginning of each slot with probability p, and the message that is sent takes all the slot.*

We choose a random slot. What is the probability of a node that runs Slotted ALOHA to send a message successfully in this slot?

Solution. We need the end host to send a message, which is probability of p, that the other nodes will not send messages in this slot: $(1-p)^{n-1}$, and that the pure ALOHA will not send for **three** slots: $(1-p)^{3n}$. Thus:

$$p_{suc} = p \cdot (1-p)^{n-1} \cdot (1-p)^{3n}$$

□

We choose a random slot. What is the probability of a node that runs Pure ALOHA to send a message successfully in this slot?

Solution. Let us divide into cases, where the message overlaps a single slot, and where it overlaps 2 slots.

- In the case where the message overlaps a single slot, the probabilities are as follows: the end host sends a message: p, the other slotted ALOHA nodes will not send any message at this slot: $(1-p)^n$, the other pure ALOHA nodes will not send any messages for two slots: $(1-p)^{2(n-1)}$, thus

$$P_{suc} = p \cdot (1-p)^n \cdot (1-p)^{2(n-1)}$$

- In the case where the message overlaps two slots, then the probabilities are: the end host will send a message: p, the other slotted nodes will not send any message for **two** slots: $(1-p)^{2n}$, the other pure ALOHA nodes will not send any messages for two slots $(1-p)^{2(n-1)}$, and thus

$$P_{suc} = p \cdot (1-p)^n\,(1-p)^{2(n-1)}$$

□

Let us now assume that the nodes that ran pure ALOHA start running slotted, with slots of length T, such that the slots are synchronised. What is the goodput of the network?

Solution. If we look at some slot (of $2T$), there are some options for successful transmissions within it:

1. Successful packet of length $2T$
2. Successful packet of length T, only in the first slot
3. Successful packet of length T at the second slot only
4. 2 successful packets of length T, one in each slot

Let us consider each case:

1. Successful packet of length $2T$: For this, we need exactly one of the n nodes running slotted ALOHA, will attempt to send:
$$\binom{n}{1} p\left(1-p\right)^{n-1} = np\left(1-p\right)^{n-1}$$
All the other nodes running pure ALOHA will not send: $\left(1-p\right)^{2n}$ Both these events are independent, so:
$$np\left(1-p\right)^{n-1}\left(1-p\right)^{2n}$$
2. Successful packet of length T, only in the first slot: For this we need exactly one node with length T to transmit:
$$\binom{n}{1} p\left(10p\right)^{n-1} = np\left(1-p\right)^{n-1}$$
All the other nodes will not send: $\left(1-p\right)^{n}$, and we need the second slot to fail, which is the inverse of one T node will send: $1-np\left(1-p\right)^{n-1}$. We multiply the independent factors and get:
$$np\left(1p\right)^{n1}\left(1p\right)^{n}\left(1np\left(1p\right)^{n1}\right)$$
3. Same as the successful packet in the second slot only
4. 2 successful packets of length T, one in each slot: For this we need exactly one of the nodes transmits a packet of length T in the first slot:
$$\binom{n}{1} p\cdot\left(1-p\right)^{n-1} = np\cdot\left(1-p\right)^{n-1}$$
We also need the same thing in the second slot, and that all the nodes that transmit $2T$ will not send: $\left(1-p\right)^{n}$. Since both events are independent, we can multiply them:
$$\left(np\left(1-p\right)^{n-1}\right)^{2}\left(1-p\right)^{n}$$

This leaves us with an overall goodput of
$$\eta = A + \frac{1}{2}B + \frac{1}{2}C + D$$
□