Tutorial 3 - ALOHA continued

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Notice: If you find any mistakes, please open an issue at `https://github.com/robomarvin1501/notes_networking`

1 Reminder

A RAP is an algorithm that is distributed across the network, and each node uses the same protocol. This is contrasted to protocols where there is a single central node that controls the communication. This is very widely used in networking protocols, since it makes adding and removing nodes quickly rather easy. This occurs at layer 2 - the data link layer. This all occurs over the shared communication channel, so a single collision domain. Should there be a collision, then all data is lost. This protocol also states that when a node broadcasts, it uses the *full* link bandwidth.

Definition 1.1 (Bandwidth). *The maximum rate of data transfer on the communication channel* $\frac{bit}{sec}$

Note, bandwidth $\neq$ propagation speed, and sometimes it refers to the width of a frequency range

Definition 1.2 (Goodput). *The fraction of the bandwidth used to send traffic **successfully**:* $\eta \in [0, 1]$

1.1 ALOHA

In general, we have $N \to \infty$ nodes in the network, a link bandwidth of B, sending L bits in a frame, with time divided into slots of length $T = \frac{L}{B}$. T is the time required for a message to be transmitted, and as we said before, successful transmissions use the entire bandwidth, and the entire time slot.

Note that a node can start transmitting *only* at the beginning of a slot. In pure ALOHA, every node is running its own time slots, they are **not** shared across the network. When a node *finishes* transmitting a message, it can detect collisions. We will discard propagation time (also known as physical delay), along with the noise on a channel.

Slotted ALOHA assumes that all node slots are synchronised, where pure ALOHA assumes that the nodes all have their own clocks, that are not necessarily synchronised.

Until now we have been considering the random variable X_p to be the number of transmission messages in a time segment T, under the binomial assumption:

$$X_p \sim Bin(n, p)$$

Today we will extend to the Poisson assumption:

$$X_p \sim Poi(gT)$$

We examined slotted and pure ALOHA under the binomial assumption last tutorial, and will be examining them both under the Poisson assumption.

2 ALOHA - Poisson Approach

A Poisson process $\{N_t | t \geq 0\}$ with a rate of g is a counting process where

- The number of events in any two disjoint intervals is independent

- The number of events in any interval of length T is a Poisson random variable with parameter gT. This depends o the interval's *length* and not its *timing*

- The sum of independent Poisson random variables is itself a Poisson random variable:

$$X \sim Poi(\lambda_1), \ Y \sim Poi(\lambda_2) \implies X + Y \sim Poi(\lambda_1 + \lambda_2)$$

Notation	Explanation	Units
g	Rate of total messages)(both new and failed) that are up for transmission	$\frac{1}{sec}$
$G = gT$	Average number of transmitted (successful and failed) messages in time interval T	None
P_{suc}	Probability of a (single) successful transmission (ie, no collision occurred)	None

Table 1: Definitions

If a message arrives in the middle of a slot, it will be sent at the beginning of the next slot.

2.1 Slotted ALOHA (Poisson) - Analysis

What is the probability for a successful frame? Recall that

$$X_p \sim Poi\left(gT\right)$$

So, we have

$$P_{t=T}\left(X=i\right) = \frac{(gT)^i}{i!}e^{-gT}$$
$$P_{suc} = P_{t=T}\left(k=1\right)$$
$$= gT \cdot e^{-gT}$$
$$= G \cdot e^{-G}$$

As for the goodput, as in the binomial model, P_{suc} is also the goodput.

The previous calculation is the perspective of the slot. Let us consider the perspective of the packet: What is the probability if a message has already been sent:

$$P_0 = P_{t=T}\left(k=0\right) = e^{-gT} = e^{-G}$$

Here, to calculate the goodput, we should consider the average number of packets transmitted in an interval T, which is by definition G:

$$\eta = G \cdot P_0 = G \cdot e^{-G}$$

We should actually calculate the conditional probability of no transmissions in period $t = T$, given that there is a transmission. However, under the assumption that $N \rightarrow \infty$, the difference between N, and $N1$ is negligible.

So, what is the maximum goodput?

$$\frac{d}{dG}\left(Ge^{-G}\right) = e^{-G} - Ge^{-G}$$
$$= e^{-G}\left(1-G\right)$$
$$\implies \frac{d}{dG}\left(\eta\right) = 0$$
$$\implies G^* = 1$$
$$\eta_{G=1} = e^{-1}$$
$$\approx 0.37$$

2.2 Pure ALOHA (Poisson) - Analysis

What is the probability for a successful packet?
Recall that given $X_p \sim Poi\left(gT\right)$, it holds that

$$P_{t=T}\left(X=i\right) = \frac{(gT)^i}{i!}e^{-gT}$$
$$P_{suc} = P_{k=0}\left(t=T\right) \cdot P_{k=1}\left(t=T\right)$$
$$= e^{-gT} \cdot gTe^{-gT}$$
$$= gT \cdot e^{-2gT}$$
$$\eta = P_{suc}$$
$$= gTe^{-2gT}$$
$$= Ge^{-2G}$$

From the packets perspective, what is the probability of successful packet? We will assume that a packet was sent, and calculate the probability that no other packet was sent in those 2 frames.

$$P_{suc} = P_{k=0}\left(t=T\right) \cdot P_{k=0}\left(t=T\right)$$
$$= e^{-gT}e^{-gT}$$
$$= e^{-2gT}$$

So, the goodput is Ge^{-2G}, and maximising for G we get

$$\frac{d}{dG}Ge^{-2G} = \frac{d}{dG}(G) \cdot e^{-2G} + G \cdot \frac{d}{dG}e^{-2G}$$
$$= e^{-2G} - 2e^{-2G}G$$
$$= e^{-2G}(1 - 2G)$$
$$e^{-2G}(1 - 2G) = 0$$
$$\implies \begin{cases} e^{-2G} = 0, & \text{never} \\ 1 - 2G = 0, & \text{if } G = \dfrac{1}{2} \end{cases}$$

The maximum goodput is

$$\eta_{max} = \frac{1}{2}e^{-\frac{2}{2}} = \frac{1}{2e}$$

3 Questions

3.1 Question 1

Messages arrive as a Poisson process with rate g. There are two types of messages in the network:

1. Messages of length s bits, with probability p

2. Message of length l bits, with probability $1 - p$

It is given that $l > s$. Every node can send only at the beginning of a slot (for both message types). The link rate is $1bit/sec$, and a slot is of length l seconds $(T = l)$. There are an infinite number of nodes.

What is the probability of a message being sent successfully? Recall that for $X_p \sim Poi(gT)$, it holds that

$$P_{t=T}(X = i) = \frac{(gT)^i}{i!}e^{-gT}$$

A successful l message will be with the probability

$$P_{suc,l} = (1 - p) \cdot P_{t=l}(k = 1) = (1 - p)gle^{gl}$$

and a successful s message will be the probability

$$P_{suc,s}p \cdot P_{t=l}(k = 1) = pgle^{-gl}$$

What is the goodput of the network?

$$\text{Define RV: } T_{suc} = \begin{cases} l, & \text{if successful } l \text{ packet} \\ s, & \text{if successful } s \text{ packet} \\ 0, & \text{else} \end{cases}$$
$$\mathbb{E}[T_{suc}] = l \cdot P_{suc,l} + s \cdot P_{suc,s}$$
$$\eta = \frac{\mathbb{E}[T_{suc}]}{l}$$
$$= \frac{l}{l} \cdot P_{suc,l} + \frac{s}{l} \cdot P_{suc,l}$$
$$= gle^{-gL}\left(\frac{s}{l}p + \frac{l}{l}(1 - p)\right)$$
$$= gle^{-gL}\left(\frac{s}{l}p + 1 - p\right)$$

3.2 Question 2

We now decide to use Pure instead of Slotted ALOHA, and we ask the same questions: Probability of a successful message, and goodput of the network.

Recall that $l > s$, and the slot length is l seconds. Other l have an unsafe start time of $(-l, l)$, where s frames have an unsafe start time of $(-s, l)$. The arrival rate of an s bit message is

$$g_s = p \cdot g$$

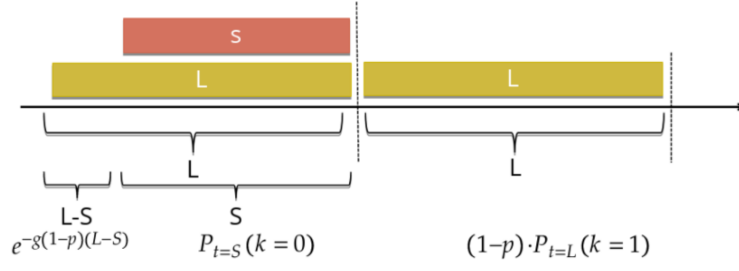and the arrival rate of an l bit message is

$$g_l = (1 - p) g$$

Figure 1: Occurrences

What is the probability that we successfully send an l message $P_{suc,l}$? The probability that no l message is sent between $-l, -s$ is

$$e^{-g_l(l-s)} = e^{-(1-p)(l-s)}$$

The probability that no l, nor s message is sent between $(-s, 0)$ is

$$P_{t=s}(k = 0) = e^{-(g_l+g_s)s} = e^{-gs}$$

The probability that exactly one l message is sent at 0 is

$$(1 - p) \cdot P_{t=l}(k = 1) = (1 - p) e^{-gl}$$

When we multiply these three independent factors, we get that

$$P_{suc,l} = (1 - p) e^{-g(2l-p(l-s))}$$

What is the probability that we successfully send an s message $P_{suc,s}$? The probability that no l message is sent between $-l, -s$ is

$$e^{-g_l(l-s)} = e^{-(1-p)(l-s)}$$

The probability that no l, nor s message is sent between $(-l, 0)$ is

$$P_t(k = 0) = e^{-(g_l+g_s)s} = e^{-gs}$$

The probability that exactly one s message is sent at 0 is

$$p \cdot P_{t=s}(k = 1) = (1 - p) e^{-gs}$$

When we multiply these three independent factors, we get that

$$P_{suc,s} = pe^{-g(l+s-p(l-s))}$$

What about the goodput? Let us define RV:

$$T_{suc} = \begin{cases} L, & \text{if successful } L \text{ packet} \\ S, & \text{if successful } S \text{ packet} \\ 0, & \text{else} \end{cases}$$

$$\mathbb{E}[T_{suc}] = L \cdot P_{suc,L} + S \cdot P_{suc,S}$$

$$\eta = \frac{L \cdot P_{suc,L}}{L} + \frac{S \cdot P_{suc,S}}{L}$$

$$= P_{suc,L} + \frac{S}{L} P_{suc,S}$$

$$= (1 - p) e^{-g(2L-p(L-S))} + \frac{S}{L} pe^{-g(L+S-p(L-S))}$$

3.3 Question 3

There are N nodes in the network, divided into 2 groups:

1. X - contains $p \cdot N$ nodes that send messages of size S, using $\frac{1}{4}$ of the bandwidth

2. Y - contains $(1 - p) N$ nodes that send messages of size $S \cdot \alpha$ such that $\alpha > 1$, using $\frac{3}{4}$ of the bandwidth

There is no overlap between the frequencies of X and Y. We will begin by discussing slotted ALOHA, every node can send only at the beginning of a slot, and $T = S$.

The arrival rate of S_α messages is

$$(1 - p) \cdot g \implies S_p^\alpha \sim Poi\left((1 - p)g\right)$$

and the arrival rate of S messages is

$$p \cdot g \implies S_p \sim Poi\left(pg\right)$$

What is the probability of a successful message? We will split into a successful S_α, and S. Recall that S_α messages are longer than a slot.
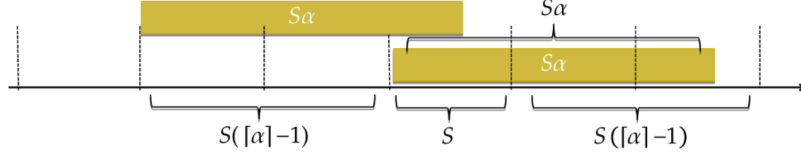

Figure 2:

$$
\begin{aligned}
P_{suc,S_\alpha} &= P_{k=0}^{S_\alpha}\left(t = S\left(\lceil\alpha\rceil - 1\right)\right) \cdot P_{k=1}^{S_\alpha}\left(t = S\right) \cdot P_{k=0}^{S_\alpha}\left(t = S\left(\lceil\alpha\rceil - 1\right)\right) \\
&= e^{-g(1-p)S(\lceil\alpha\rceil - 1)} \cdot g\left(10 - p\right)Se^{-g(1-p)S} \cdot e^{-g(1-p)S(\lceil\alpha\rceil - 1)} \\
&= g\left(1 - p\right)Se^{-g(1-p)S(2\lceil\alpha\rceil - 1)}
\end{aligned}
$$

and

$$
\begin{aligned}
P_{suc,S} &= P_{k=1}^{S}\left(t = S\right) \\
&= gpSe^{-gpS}
\end{aligned}
$$

What about the goodput? We are looking at some time window (of length $S\lceil\alpha\rceil$). During this window, there are some cases:

- A successful packet that starts at t, and contributes $S\alpha$

- A successful packet that starts at $t - 1$, and contributes $S\left(\alpha - 1\right)$

- $\vdots$

- A successful packet that starts at $t + S$, and contributes $S\left(\lceil\alpha\rceil - 1\right)$

So we can construct a very large random variable

$$
T_{suc,S\alpha} = \begin{cases}
S\alpha, & \text{if successful packet that starts at } t \\
S\left(\alpha - 1\right), & \text{if successful packet that starts at } t - S \\
S\left(\alpha - 2\right), & \text{if successful packet that starts at } t - 2S \\
\vdots & \\
S\left(\lceil\alpha\rceil - 1\right), & \text{if successful packet that starts at } t + S \\
S\left(\lceil\alpha\rceil - 2\right), & \text{if successful packet that starts at } t + 2S \\
\vdots & \\
S \cdot 1, & \text{if successful packet that starts at } t + S\left(\lceil\alpha\rceil - 1\right) \\
0, & \text{else}
\end{cases}
$$

So

$$
\begin{aligned}
\mathbb{E}\left[T_{suc,S\alpha}\right] &= P_{suc,S\alpha}\left(S\alpha + \sum_{k=1}^{\lceil\alpha\rceil - 1} S\left(\alpha - k\right) + \sum_{k=1}^{\lceil\alpha\rceil S(\lceil\alpha\rceil - k)}\right) \\
&= P_{suc,S\alpha}S\left(\alpha + \sum_{k=1}^{\lceil\alpha\rceil - 1}\alpha - \sum_{k=1}^{\lceil\alpha\rceil - 1}k + \sum_{k=1}^{\lceil\alpha\rceil - 1}k\right) \\
&= P_{suc,S\alpha}S\left(\alpha + \left(\lceil\alpha\rceil - 1\right)\alpha\right) \\
\mathbb{E}\left[T_{suc,S\alpha}\right] &= P_{suc,S\alpha}S\alpha\lceil\alpha\rceil
\end{aligned}
$$

We will then do the same thing with S.

In total, we get

$$T_{suc,S\alpha} = \begin{cases} S, & \text{if successful } S \text{ packet} \\ 0, & \text{else} \end{cases} \implies \mathbb{E}\left[T_{suc,S}\right] = S \cdot P_{suc,S}$$

So the goodput of the network comes out to be

$$\begin{aligned} \eta &= \frac{3}{4}\eta_{S\alpha} + \frac{1}{4}\eta_S \\ &= \frac{3}{4}\frac{\mathbb{E}\left[T_{suc,S\alpha}\right]}{S\lceil\alpha\rceil} + \frac{1}{4}\frac{\mathbb{E}\left[T_{suc,S}\right]}{S} \\ &= \frac{3}{4}\frac{S\alpha\lceil\alpha\rceil \cdot P_{suc,S\alpha}}{S\lceil\alpha\rceil} + \frac{1}{4}\frac{S \cdot P_{suc,S}}{S} \\ &= \frac{3}{4}\alpha P_{suc,S\alpha} + \frac{1}{4}P_{suc,S} \end{aligned}$$